

Five-dimensional gravity with time-dependent warp factor

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We have considered gravity in a five-dimensional warped product space-time, with a time-dependent warp factor and a time-dependent extra dimension. The field equations have been solved for a spatially flat FRW brane and the energy conditions and the nature of bulk geometry have been examined. At low energies, the trapping of fields within the brane implies a correlation between the warp factor and the extra-dimensional scale factor. Generally, the bulk is not conformally flat. At high energies, the bulk is sourced by a dilaton-like minimally coupled scalar field, which depends on the warp factor. Both the bulk and the brane are found to be Anti de Sitter, with some exceptions.

I. INTRODUCTION

The theories of particle physics beyond the standard model [1–4], taught us that the universe is possibly a (3+1)-dimensional spacetime embedded in a higher-dimensional bulk, with the extra-dimensions playing a crucial role in a dynamical spacetime. The knowledge that the gauge fields may consistently appear on the ten-dimensional (10D) boundary of a Z_2 -symmetric eleven-dimensional (11D) space-time [5] and that the effective five-dimensional (5D) theory of the strongly coupled heterotic strings is a gauged version of the ($N = 1$) 5D supergravity with four-dimensional (4D) boundaries [6], have led to extensive studies in this area. The simplest illustration can be given in the framework of a 5D theory with a single warped extra dimension [7–9]. The issue of localization has been addressed by many [10–12]. Cosmological consequences have been studied in the M-theory models with two branes at finite distances apart, acting as the boundaries of the bulk spacetime [13, 14], and in the Randall-Sundrum (RS) model [10] of both the single brane and two brane versions [15–19].

The bulk is "slightly warped" in the scenarios of heterotic M theory with the inter-brane distances being represented by scalar fields (called *moduli*), introduced through the integration constants in the classical solutions. For solutions existing over a continuous range of integration constants, the corresponding moduli are massless, giving rise to long range scalar forces, which tend to evolve cosmologically, being time-dependent. This leads to time-variations of the fundamental constants [20], indicating that at least some moduli fields are not stabilized. To avoid this, the moduli fields are assumed to be stabilized by some mechanism in the early universe [21]. On the other hand, the RS model is "highly warped" and without any bulk scalar (except for the cosmological constant), with matter fields on the brane being responsible for the dynamics of the brane. However, string theory requires the dimensionally reduced low energy effective action in 5D, to include both the higher-order curvature terms, as well as dilatonic scalar fields [6, 22]. Thus scalar fields were introduced in the bulk of the RS model to stabilize the scale of the extra dimension [23], with the single modulus, called the "radion", being related to the thickness of the AdS slice. The resulting 4D theories at low energies in both the M theory and RS models are very similar to the scalar-tensor theories [12, 24, 25].

In the RS1 setup without any bulk scalar, the weak gravity at low energy on either brane is Brans-Dicke type. The inter-brane distance is arbitrary and the corresponding mode is massless, thereby playing the role of a Brans-Dicke scalar. In presence of a bulk scalar field which couples to the branes, the inter-brane distance is no longer arbitrary and the expected massless mode acquires a nonzero mass, so that Einstein gravity is recovered [24, 26]. In the RS2 setup, although, the hierarchy problem cannot be solved, there are several important consequences [27–29]. One of them is an alternative scenario of the inflationary universe, where slow-roll inflation on the brane is driven solely by the dynamics of a dilaton-like minimally coupled scalar field living in the 5D bulk, and not by an inflaton field. The 5D effective action includes scalar fields of gravitational origin, having a suitable potential and Newtonian gravity is recovered in the 4D effective theory. The bulk scalar also facilitates to generate the hypersurface in the form of the so-called thick brane [30].

In the RS model, the bulk and the branes are maximally symmetric. The fields living on the negative tension brane, have masses proportional to the radion expectation value and for generic matter in the bulk, the one loop effective potential is not sufficient to stabilize the radion at a phenomenologically acceptable mass [31]. In the warped braneworld models which are not maximally symmetric, the bulk contains a dilaton field, acting as the inflaton field, with an exponential potential coupled to the 5D gravity. The solutions have a power-law warp factor $a(y) \propto y^q$, where y is the distance along the extra dimension and q is determined by the parameters in the Lagrangian. This yields a power-law inflation on the brane [32, 33], although the bulk itself is not inflating. The RS model is recovered in the

limit $q \rightarrow \infty$. The back reaction of the dilaton coupling in the bulk to the 4D brane gives rise to the non-vanishing component of the energy-momentum tensor along the fifth dimension [34].

Many authors have constructed various generalized and time-dependent solutions in an attempt to develop realistic models which can address the unsolved issues in astrophysics and cosmology [35–41]. Cosmological compactifications on Einstein spaces of negative curvature, have yielded accelerating 4D FLRW cosmologies [38]. The no-go theorem forbidding acceleration in the time-independent compactifications of string theory, are not applicable to a general class of warped supergravity models that allows at least one noncompact extra dimension [42]. The de Sitter solutions of warped supergravity have finite 4D Newton’s constant and the 4D massless graviton wave function is normalizable on an inflating brane (FLRW universe). Recently, dynamical brane solutions in supergravity and in string theory have also been studied [43]. Depending on the ansatz for the fields, the warp factors in the solutions can be time-dependent. After compactification of the internal space, such solutions lead to the FLRW cosmology with power-law expansion. However, the power of the scale factor is too small for the solutions to yield a realistic expansion law. This opens up the possibility of further investigations with generalized spacetime geometries.

With this aim, we have considered a thick brane [44] embedded in an AdS bulk, in presence of a dilaton-like bulk gravitational scalar which is time-varying, a general exponential warp factor which depends both on time as well as on the extra-dimensional coordinate (a special type was considered in [39]) and a time-dependent extra dimension. During the early stages of cosmological evolution (the high energy regime), the universe underwent a very rapid phase of expansion, so that the process of localization of gravity was presumably time-dependent. This can be described by a time-dependent warp factor, which in turn gives rise to a time-varying bulk cosmological constant, with the scale of the extra dimension also time-dependent. The gravitational force law was sensitive to the background cosmological expansion [27], and the 4D Newton’s constant was time-varying. Gravity, along with scalar fields and particles could access the extra dimensions [45]. The effective cosmological constant of the hypersurface is determined by the curvature of the bulk metric, which is time-varying due to the time-dependent process of localization, and this curvature controls the 4D Newton’s constant. The dynamical evolution of the brane is determined by the Einstein equations for the combined brane-bulk system and the dynamics of the gravitational field is modified by the presence of the time-dependent warp factor. If this modification takes place at today’s Hubble scale, H_0 , then it is expected that this will alter the gravitational force law at distance scales much smaller than H_0^{-1} [46], leading to an expansion history that can be identically reproduced by a dynamical dark energy model. Somewhere around the end of inflation, the time-dependent process of localization came to an end, leading to the finite 4D Newton’s constant in the present universe. This can be related to the infinite extent of the fifth dimension and is possible if the universe is inflating. The warp factor is regular at the boundary of the 3-brane and is well-behaved at infinite distance from the brane. The brane tension is induced both by the bulk cosmological constant, as well as the curvature related to the expansion of the (3+1) spacetime and the physical universe eventually accelerates. For an AdS bulk, the zero-mode graviton may not be localized on the brane [47].

Having presented the mathematical preliminaries in Section II, we construct the five-dimensional field equations in Section III and examine the status of the different energy conditions in the bulk. To get a clear picture of the geometry and the corresponding physical implications, we have considered a number of toy models. We have considered an isotropic bulk and have determined the nature of the bulk matter and that of the warp factor. To determine the exact analytic solutions, we have assumed a definite correlation between the warping function and the extra-dimensional scale factor, following the results of supersymmetric brane models [48]. In Section IV, we assumed a vanishing flux term for the generalized metric, signifying confinement of matter or energy (or both) within the 4D hypersurface, and have investigated the possibility of having the bulk as a conformally flat spacetime and the requirements for a stabilized bulk. In Section V, we consider the flux term to be non-zero, which therefore represents the condition when gravity becomes essentially higher dimensional. In this regime, we assume that exotic matter in the form of a dilaton scalar (minimally coupled to gravity) propagates in the bulk along with gravitons. We have considered two types of scalar fields: a massless non-self-interacting scalar field, for which the solutions are obtained for various types of warping function, and a scalar field with self-interaction potential for the most general warped metric in 5D, for which the solutions are obtained and the nature of cosmology in the bulk and the brane are analysed. In Section VI, we have examined the effect of placing the brane within the bulk and have discussed the nature of localization of gravitons and other matter fields, in brief. The summary and discussions have been presented in Section VII.

II. MATHEMATICAL PRELIMINARIES

Let us consider the generalized 5D action in presence of a bulk scalar field with potential, minimally coupled to gravity, as follows [41, 46, 49, 50]

$$S = \int d^5x \sqrt{g} \left[\frac{1}{2\bar{\kappa}^2} (\bar{R} - 2\bar{\Lambda}) + \frac{1}{2} g^{AB} \nabla_A \psi \nabla_B \psi - V(\psi) \right] + \int d^4x \sqrt{-g} L_m, \quad (1)$$

supplemented with the brane curvature term

$$M_P^2 \int d^4x \sqrt{-g} R, \quad (2)$$

where \bar{g}_{AB} is the 5D metric of signature (+ - - -), $\bar{\Lambda}$ is the bulk cosmological constant and \bar{R} is the 5D scalar curvature. The first term in (1) corresponds to the 5D Einstein-Hilbert action in presence of the bulk scalar. The Lagrangian density L_m represents all other contribution to the action which are not strictly gravitational, including the contribution of the matter fields localized on the brane and any interaction between the brane and the bulk. The constant $\bar{\kappa}$ is related to the 5D Newton's constant $G_{(5)}$ and the 5D reduced Planck mass $M_{(5)}$ by the relation

$$\bar{\kappa}^2 = 8\pi G_{(5)} = M_{(5)}^{-3}. \quad (3)$$

The dynamics of the 5D space-time, including that of the 4D hypersurface representing the observed universe, is determined by the 5D field equations

$$\bar{G}_{AB} = -\bar{\Lambda}\bar{g}_{AB} + \bar{\kappa}^2 \bar{T}_{AB} \quad (4)$$

where \bar{G}_{AB} is the 5D Einstein tensor and \bar{T}_{AB} represents the 5D energy-momentum tensor. For a generalized 5D metric ansatz of the type as in [15], there may exist definite correlations between the various scale factors in the line element. A specific correlation (viz. $n^2 = a^2 = e^{2f}$) leads us to the so-called "warped spacetime" [51]. Study of such spacetimes is important in theories of gravity, since they constitute a wide variety of exact solutions to the field equations. Such warped spacetimes has been used to solve the hierarchy problem and to recover Newtonian gravity in the low energy effective theory [10]. We therefore choose a 5D metric ansatz given by

$$dS^2 = e^{2f(t,y)} (dt^2 - R^2(t)(dr^2 + r^2 d\theta^2 + r^2 \sin(\theta)^2 d\phi^2)) - A^2(t,y) dy^2 \quad (5)$$

where y is the coordinate of the fifth dimension, t denotes the conformal time and the function $A(t,y)$ parametrizes the scale of the extra dimension at different times and at different locations in the bulk. The "warping function", f , is a smooth function and $e^{2f(t,y)}$ is the time-dependent warp factor. The observed universe is represented by the hypersurface at $y = y_0$. The complicated nature of the field equations owing to the generalized metric makes it necessary to impose additional constraints on the geometry to determine the solutions.

The RS scenario led to unnatural correlation between the bulk cosmological constant and the brane tensions. However, brane-bulk supersymmetry correctly correlates the brane tensions and the bulk cosmological constant in the supersymmetric RS models [48]. Under this condition, the metric (5) will be a solution of the 5D Einstein equations, *with the warp factor being related to the extra-dimensional scale factor*. For a stabilized bulk, $A(t,y)$ must settle down to a fixed value. A stabilized bulk with constant curvature and characterized by a negative cosmological constant is free from the appearance of nonconventional cosmologies [52]. For a conformally flat bulk (a space of constant curvature), the extra-dimensional scale factor can be a growing function of time [33]. For such a bulk, the resulting field equations are much simpler owing to the simplified geometry. However, a generalized bulk may not be conformally flat. A spatially flat FRW-type braneworld is appropriate for achieving cosmological solutions and can be embedded in any constant curvature bulk [53], in spite of representing a dynamical brane. In our model, we extend it to the case of any arbitrary bulk which contains the present universe embedded in it and evolving with time.

III. FIELD EQUATIONS AND ENERGY CONDITIONS IN THE BULK

The non-vanishing components of the 5D Einstein tensor for the space-time under consideration are

$$\bar{G}_t^t = \frac{3}{e^{2f}} \left(\frac{\dot{R}^2}{R^2} + \frac{2\dot{R}\dot{f}}{R} + \dot{f}^2 + \frac{\dot{f}\dot{A}}{A} + \frac{\dot{R}\dot{A}}{R A} \right) - \frac{3}{A^2} \left(2f'^2 + f'' - \frac{f'A'}{A} \right), \quad (6)$$

$$\bar{G}_y^t = -\frac{3}{e^{2f}} \left((\dot{f})' - \frac{\dot{A}}{A} f' \right), \quad (7)$$

$$\bar{G}_t^y = \frac{3}{A^2} \left((\dot{f})' - \frac{\dot{A}}{A} f' \right), \quad (8)$$

$$\bar{G}_y^y = \frac{3}{e^{2f}} \left(\frac{\ddot{R}}{R} + \frac{\dot{R}^2}{R^2} + \frac{3\dot{R}\dot{f}}{R} + \dot{f}^2 + \ddot{f} \right) - \frac{6f'^2}{A^2}, \quad (9)$$

and

$$\bar{G}_I^I = \frac{1}{e^{2f}} \left(\frac{2\ddot{R}}{R} + \frac{\dot{R}^2}{R^2} + \frac{4\dot{R}\dot{f}}{R} + \dot{f}^2 + 2\ddot{f} + \frac{\dot{f}\dot{A}}{A} + \frac{2\dot{R}\dot{A}}{R A} + \frac{\ddot{A}}{A} \right) - \frac{3}{A^2} \left(2f'^2 + f'' - \frac{f'A'}{A} \right). \quad (10)$$

Above, a dot represents differentiation with respect to the conformal time t and a prime stands for differentiation with respect to the fifth coordinate y and $I = r, \theta, \phi$. The components of the bulk stress-energy tensor are given by

$$\bar{T}_t^t = \bar{\rho}, \quad \bar{T}_j^i = -\bar{P}, \quad \bar{T}_y^y = -\bar{P}_y, \quad \bar{T}_y^t = \bar{Q}.$$

where $\bar{\rho}$, \bar{P} and \bar{P}_y are the energy density and pressure in the bulk, with \bar{Q} representing a non-trivial flux term. The non-zero \bar{G}_y^t term with the corresponding \bar{T}_y^t component, indicates a flux of matter or energy (or both) from the 4D hypersurface into the bulk along the extra dimension. The energy scale necessary to ensure such an access is few hundred GeV. For such a generic case of the bulk, the corresponding energy-momentum tensor may not be diagonalizable, although for "physically reasonable" matter content on the hypersurface [54] (and may be in the bulk), we expect the stress-energy tensor to be diagonalizable. In the previous work [39], it was found that the validity of the strong energy condition (SEC) in the bulk is governed by the nature of warping, as well as by the effect of the extra dimension. Keeping in mind the possibility of a negative pressure component of matter-energy (dark energy or cosmological constant), we require that at least the weak energy condition (WEC) and the null energy condition (NEC), must be obeyed. This implies that we must have

$$\bar{\rho} \geq 0 \quad \text{and} \quad \bar{\rho} + \bar{P}_i \geq 0 \quad (i = 1, 2, 3, 4). \quad (11)$$

Assuming $8\pi G_{(5)} = 1$ in the remaining analysis, we obtain the field equations in 5D as

$$\bar{G}_t^t = -\bar{\Lambda} + \bar{\rho}, \quad (12)$$

$$\bar{G}_J^I = -\bar{\Lambda} - \bar{P}, \quad (13)$$

$$\bar{G}_y^y = -\bar{\Lambda} - \bar{P}_y, \quad (14)$$

and

$$\bar{G}_y^t = -\bar{\Lambda} - \bar{Q}. \quad (15)$$

The complicated nature of the various components of the Einstein tensor makes it difficult for us to comment directly on the status of the energy conditions from the above equations. To proceed further, let us write down the energy conditions in terms of the energy eigen values (say, a_i), which must be real for matter-energy obeying the WEC [55]. These eigen values are given by the roots of the characteristic equation for the energy-momentum tensor:

$$\det(\bar{T} - aI) = 0. \quad (16)$$

This implies that

$$\begin{vmatrix} -\bar{\rho} + a & 0 & 0 & 0 & \bar{Q} \\ 0 & \bar{P} + a & 0 & 0 & 0 \\ 0 & 0 & \bar{P} + a & 0 & 0 \\ 0 & 0 & 0 & \bar{P} + a & 0 \\ \bar{Q} & 0 & 0 & 0 & \bar{P}_y + a \end{vmatrix} = 0.$$

Solving this equation, we find that the eigen values are given by

$$a_{0,4} = \frac{1}{2} \left[(\bar{\rho} - \bar{P}_y) \pm \sqrt{(\bar{\rho} + \bar{P}_y)^2 - 4\bar{Q}^2} \right], \quad (17)$$

$$\text{and} \quad a_1 = a_2 = a_3 = \bar{P}. \quad (18)$$

For these eigen values to be real, we must have,

$$(\bar{\rho} + \bar{P}_y)^2 - 4\bar{Q}^2 \geq 0, \quad (19)$$

along with the following restriction, so as to satisfy (11), i.e.

$$a_0 \geq 0 \quad \text{and} \quad a_0 - a_i \geq 0 \quad (i = 1, 2, 3, 4). \quad (20)$$

The conservation of the energy-momentum tensor $\bar{T}_b^a; a = 0$ leads us to the two equations

$$\dot{\bar{\rho}} + 3(\bar{\rho} + \bar{P}) \left(\dot{f} + \frac{\dot{R}}{R} \right) + (\bar{\rho} + \bar{P}_y) \frac{\dot{A}}{A} = 0, \quad (21)$$

and

$$\bar{P}'_y + (\bar{\rho} - 3\bar{P} + 4\bar{P}_y)f' = 0. \quad (22)$$

where $\frac{\dot{R}}{R} = H$, is the Hubble parameter on the brane.

A. Solutions for an isotropic bulk

For an isotropic bulk, $\bar{P} = \bar{P}_y$, so that the limiting case for condition (19) reduces to

$$(\bar{\rho} + \bar{P}) = |2\bar{Q}|, \quad (23)$$

where \bar{Q} may be positive or negative but $(\bar{\rho} + \bar{P})$ must be non-negative so as not to violate the WEC. Assuming positive flux for \bar{Q} , we can rewrite equation (21) in the form

$$\dot{\bar{\rho}} = -2\bar{Q} \left(3\dot{f} + \frac{3\dot{R}}{R} + \frac{\dot{A}}{A} \right), \quad (24)$$

which represents the equation of continuity for this isotropic bulk. For a surface enclosing a given volume of the bulk (such that it encloses the 4D universe in the form of a 3-brane), since \bar{Q} is positive, the energy density inside this volume should decrease with time, if the quantity inside the brackets is positive. Thus, the rate of decrease of energy density also depends on the rate of variation of the warp factor, the rate of expansion of the observed universe and the normalized rate of variation of the extra-dimensional scale factor. From (22) we get

$$(\bar{\rho} + \bar{P}) = -\frac{\bar{P}'}{f'}. \quad (25)$$

The l.h.s. of the above equation will be non-negative under the following possibilities:

- i $\bar{P}' = 0$, which implies that the pressure is a constant. This is possible when the bulk matter energy-momentum tensor is that of a perfect fluid representing dust. We can have f' to be either positive or negative, which means that the warp factor may be either growing or of decaying type. However, the decaying type is more favorable for cosmologically relevant solutions [39].
- ii $\bar{P}' > 0$, for which f' should necessarily be negative i.e. the warp factor is of decaying type, so that the WEC is valid.
- iii $\bar{P}' < 0$, in which case f' is positive and the warp factor is of growing type.

Combining equations (13) and (14) for this isotropic bulk we obtain the relation

$$\frac{1}{e^{2f}} \left(\frac{\ddot{R}}{R} + \frac{2\dot{R}^2}{R^2} + \frac{5\dot{R}\dot{f}}{R} + 2\dot{f}^2 + \ddot{f} - \frac{\dot{f}\dot{A}}{A} - \frac{2\dot{R}\dot{A}}{R A} - \frac{\ddot{A}}{A} \right) = -\frac{3}{A} \left(\frac{\partial}{\partial y} \left(\frac{f'}{A} \right) \right). \quad (26)$$

Since the l.h.s. involves only time-derivatives and the r.h.s. only the derivative with respect to the fifth coordinate, without any loss of generality, we can assume that they are separately equal to some constant (say, C). From the r.h.s. we can infer that the variation of the extra-dimensional scale factor along the direction of the extra coordinate at different locations in the bulk, depends on the variation of the warp factor along the extra dimension. However, accurate predictions can only be made if we know the exact functional relations.

Equating the constant C to zero, we obtain the following relation from the l.h.s. of (26)

$$\left(\frac{\ddot{R}}{R} + \frac{2\dot{R}^2}{R^2} + \frac{5\dot{R}\dot{f}}{R} + \dot{f} \left(2\dot{f} - \frac{\dot{A}}{A} \right) + \ddot{f} - \frac{2\dot{R}\dot{A}}{R A} - \frac{\ddot{A}}{A} \right) = 0. \quad (27)$$

B. A specific relation between f and A in case of the isotropic bulk

Remembering that the warp factor may be related to the extra-dimensional scale factor, we assume that

$$2\dot{f} = \frac{\dot{A}}{A} = B(t, y). \quad (28)$$

In that case the relation between f and A may be of the type $f = \frac{1}{2} \ln A + \text{constant}$.

1. A de Sitter brane

For a dS 3-brane, the scale factor $R(t)$ is given by

$$R(t) \sim e^{Ht}. \quad (29)$$

The l.h.s. of (26) can now be simplified further to yield:

$$\dot{H} + 3H^2 + H\dot{f} + \ddot{f} - \frac{\ddot{A}}{A} = 0, \quad (30)$$

and finally we have

$$2\dot{H} + 6H^2 + HB - 2B^2 - \dot{B} = 0. \quad (31)$$

A straightforward solution of (33) is obtained when B is simply a function of time. In such a case, the spacetime is unwarped, with the extra-dimensional scale factor depending only on time. However, such type of solutions destroy the generalized feature of the spacetime geometry that we intend to investigate. The other way round is to try with various explicit functional form of B and see which one yields a physically consistent result.

2. A radiative brane

The scale factor for a radiative brane is

$$R(t) \sim \sqrt{t}. \quad (32)$$

Using (28), we have from (27)

$$\left(\frac{\ddot{R}}{R} + \frac{2\dot{R}^2}{R^2} + \frac{\dot{R}\dot{f}}{R} - \ddot{f} - 4\dot{f}^2 \right) = 0. \quad (33)$$

Substituting (32) in (33), we get

$$\frac{\ddot{f} + 4\dot{f}^2}{1 + 2t\dot{f}} = \frac{1}{4t^2}. \quad (34)$$

The above gives the nature of the time-variation of the warping function at a given location in the bulk, specified by $y = \text{constant}$. For a static brane, $y = \text{constant}$ is sufficient to specify the position of the brane along the extra dimension. However, for a dynamic brane, the deviation of the hypersurface from the tangent plane will change with time [39, 53]. This in turn will monitor the nature of the localization of fields on the brane. Since the warp factor is smooth, we assume that the deviation of the hypersurface changes slowly with time. During the epoch when, $\ddot{f} = 0$, (34) may be simplified further to yield

$$\dot{f} = \frac{\alpha}{t}, \quad (35)$$

where $\alpha = (2 \pm \sqrt{68})/32$, leading to a warping function of the type

$$f \sim \ln(t^\alpha \cosh(cy)), \quad (36)$$

with the extra-dimensional scale factor as

$$A \sim t^\alpha \cosh(cy). \quad (37)$$

This is just a particular solution, for which the warp factor is of product type. The general solution when $\ddot{f} \neq 0$ and with the deviation varying rapidly, will be more complicated.

IV. LOW ENERGY GRAVITY

We know that at low energies, gravity is localized at the brane along with particles. To prevent matter or energy flowing out of the brane along the fifth dimension, we require the flux term to be zero, i.e. $\bar{T}_y^t = 0$, which implies that $\bar{G}_y^t = 0$ and hence we obtain

$$\dot{f}' = \frac{\dot{A}}{A} f'. \quad (38)$$

Assuming that both A , f and their first order derivatives are continuous, (17) can be easily integrated to give the result

$$A(t, y) = \chi(y) f'(t, y). \quad (39)$$

Thus the *extra-dimensional scale factor is indeed related to the warp factor*. The extra-dimensional scale factor at a given y and t , depends on the way the warping function varies along the extra dimension at that instant at the given location and will be different at different locations in the bulk. The manner in which gravity is localized on the 4-dimensional hypersurfaces at different locations in the bulk (being monitored by the warp factor), will also be different at different times.

Consequently, the Einstein equations for the above spacetime get reduced to the form

$$\bar{\rho} - \Lambda_{(5)} = \frac{3}{e^{2f}} \left(\frac{\dot{R}}{R} + \dot{f} \right) \left(\frac{\dot{R}}{R} + \dot{f} + \frac{\dot{f}'}{f'} \right) - \frac{3}{\chi^2 f'} \left(2f' - \frac{\chi'}{\chi} \right), \quad (40)$$

$$\bar{P} + \Lambda_{(5)} = -\frac{1}{e^{2f}} \left(2 \left(\frac{\ddot{R}}{R} + \frac{\dot{R}\dot{f}}{R} + \ddot{f} \right) + \left(\frac{\dot{R}}{R} + \dot{f} \right)^2 + \frac{\dot{f}'}{f'} \left(\dot{f} + \frac{2\dot{R}}{R} \right) + \frac{\ddot{f}'}{f'} \right) + \frac{3}{\chi^2 f'} \left(2f' - \frac{\chi'}{\chi} \right), \quad (41)$$

and

$$\bar{P}_y + \Lambda_{(5)} = -\frac{3}{e^{2f}} \left(\frac{\ddot{R}}{R} + \frac{\dot{R}^2}{R^2} + \frac{3\dot{R}\dot{f}}{R} + \dot{f}^2 + \ddot{f} \right) + \frac{6}{\chi^2}. \quad (42)$$

Thus we have six unknowns (f , R , χ , $\bar{\rho}$, \bar{P} and \bar{P}_y) and three independent equations. To proceed further, we need to impose additional constraints to solve the field equations.

A. Possibility of a conformally flat bulk

The nonzero components of the Weyl tensor for the spacetime under the condition $A(t, y) = \chi(y) f'(t, y)$, are given by

$$C_{AB}^{AB} = \text{constant} \times \left(\frac{1}{e^{2f}} \left(\frac{\ddot{R}}{R} - \frac{\dot{R}}{R} \left(\dot{f} + \frac{\dot{R}}{R} \right) + \ddot{f} - \dot{f}^2 + \frac{\dot{f}'}{f'} \left(\frac{\dot{R}}{R} + 2\dot{f} \right) - \frac{\ddot{f}'}{f'} \right) \right) \quad (43)$$

where, only the constant factor varies for the different components. For this bulk to be conformally flat, we need the following condition to be satisfied:

$$f(t, y) = F_1(t) F_2(y) \quad (44)$$

where F_1 and F_2 are arbitrary functions such that

$$\frac{dF_2(y)}{dy} = 0. \quad (45)$$

This implies that the conformally flat solutions have $f(t, y) = (F_1(t) \times \text{constant})$ and hence $A(t, y) = \chi(y) f'(t, y) = 0$. Therefore, we cannot have a conformally flat bulk with such a generalised scale factor for the extra dimension, although we obtain the conformally flat 5D bulk if the extra-dimensional scale factor is given by $A = A(t)$, which may be a growing function of time [33], corresponding to a moving brane within a static bulk.

1. A particular case

We now examine a very special type of warped metric for which $f(t, y) = \ln(at + b\cosh(cy))$, such that

$$e^{2f(t,y)} = A^2(t, y) = (at + b\cosh(cy))^2. \quad (46)$$

In this case, the Einstein tensors for the bulk and its 4D part are diagonal, and the bulk turns out to be conformally flat with identical components for the Riemann tensor if the following condition is valid:

$$\frac{\ddot{R}}{R} - \frac{\dot{R}^2}{R^2} = 0. \quad (47)$$

Under this condition, the Hubble parameter of the brane turns out to be a constant. The Weyl tensor of the 4D part vanishes identically. The Ricciscalar for the 5D bulk, and for the corresponding 4D part are identical and is given below:

$$R = -\frac{6}{(at + b\cosh(cy))^2} \left(\frac{\ddot{R}}{r} + \frac{\dot{R}^2}{R^2} \right), \quad (48)$$

indicating that both the 4D part and the bulk are AdS, provided $\frac{\dot{R}^2}{R^2} > |\frac{\ddot{R}}{r}|$. The bulk pressure is isotropic along the three spatial directions on the brane, but is different along the extra dimension. The 4D part itself is isotropic. The scale factor for the 3-brane is then given by $R(t) = ae^{Ht}$, and its deceleration parameter is given by the unique value

$$q = -1 \quad (49)$$

representing a universe in an epoch of uniform acceleration.

B. The case of a stabilized bulk

For this condition, we must have $\dot{A} = 0$, so that $A = A(y)$. Re-scaling $A(y) \rightarrow 1$, we find that for $\bar{T}_y^t = 0$, we need $\dot{f}' = 0$. Thus either $f = f(t)$ or $f = f(y)$.

1. $f = f(y)$:

This corresponds to the usual RS scenario. For $f'' = 0$, we have $f = \text{const} \times y$, which represents a dS 3-brane embedded within a 5D AdS. When the bulk is of constant curvature, the bulk Weyl tensor vanishes and the bulk is characterized by a negative cosmological constant with $\rho_B = -P_B$.

2. $f = f(t)$:

This is the unwarped case. We can absorb $f(t)$ inside the 4D metric.

3. $f(t, y) = \frac{1}{2}(\ln\tau(t) + \ln\Gamma(y))$:

This is a particular case where the warp factor is taken in the product form and has been discussed earlier [39]. The assumption was made in order to transform the Einstein equations into a simpler form, which can be solved by some straightforward calculations. The energy-momentum tensor has only the \bar{T}_y^y component other than the components on the brane. As mentioned before, the validity of SEC in the bulk for this metric is governed by the nature of warping and the effect of the extra dimension. For an isotropic bulk, with $\tau(t) = t^{-1/2}$ and $R(t) = t^{1/2}$, the WEC is valid. The 4D universe is initially decelerated, but makes a transition to an accelerated phase at later times, leading to an interpretation of dynamical dark energy.

V. GRAVITY AT HIGH ENERGIES

Here the energy scale is in the GeV range and particles can escape into the extra dimension. We shall consider two specific cases of the bulk scalar field, namely an ordinary, non-self-interacting massless scalar field and a scalar field with potential.

A. A massless scalar field in the bulk

The energy-momentum tensor for a non-self-interacting massless scalar field source $\psi(t, y)$ [41] is given by

$$\bar{T}_{IJ}^{scalar} = -\partial_I \psi \partial_J \psi + \frac{1}{2} g_{IJ} \partial_K \psi \partial^K \psi \quad (50)$$

Case 1:

Here we assume the warping function $f(t, y)$ to be given by

$$f(t, y) = \frac{1}{2} (\ln \tau(t) + \ln \Gamma(y)). \quad (51)$$

With this choice, the 5D metric becomes

$$dS^2 = \tau(t) \Gamma(y) (dt^2 - R^2(t) (dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2)) - A^2(t, y) dy^2. \quad (52)$$

The difference from [39] is in the functional form of the extra-dimensional scale factor. The components of the Einstein tensor for the bulk are now

$$\bar{G}_{tt} = \frac{3\bar{g}_{tt}}{4\tau\Gamma} \left(\frac{4\dot{R}\dot{\tau}}{R\tau} + \frac{4\dot{R}^2}{R^2} + \frac{\dot{\tau}^2}{\tau^2} + \frac{2\dot{\tau}\dot{A}}{\tau A} \right) - \frac{6\bar{g}_{tt}}{4A^2} \left(\frac{\Gamma''}{\Gamma} - \frac{\Gamma' A'}{\Gamma A} \right), \quad (53)$$

$$\bar{G}_{ty} = \frac{3\dot{A}\Gamma'}{2A\Gamma} \quad (54)$$

$$\bar{G}_{yy} = \frac{3\bar{g}_{yy}}{4\tau\Gamma} \left(\frac{2\ddot{\tau}}{\tau} - \frac{\dot{\tau}^2}{\tau^2} + \frac{6\dot{R}\dot{\tau}}{R\tau} + \frac{4\dot{R}^2}{R^2} + \frac{4\ddot{R}}{R} \right) - \frac{6\bar{g}_{yy}}{4A^2} \left(\frac{\Gamma'}{\Gamma} \right)^2, \quad (55)$$

$$\bar{G}_{ij} = \frac{\bar{g}_{ij}}{4\tau\Gamma} \left(\frac{4\ddot{\tau}}{\tau} - \frac{3\dot{\tau}^2}{\tau^2} + \frac{8\dot{R}\dot{\tau}}{R\tau} + \frac{4\dot{R}^2}{R^2} + \frac{8\ddot{R}}{R} + \frac{2\dot{\tau}\dot{A}}{\tau A} + \frac{8\dot{R}\dot{A}}{RA} + \frac{4\ddot{A}}{A} \right) - \frac{6\bar{g}_{ij}}{4A^2} \left(\frac{\Gamma''}{\Gamma} - \frac{\Gamma' A'}{\Gamma A} \right). \quad (56)$$

The components of the bulk energy-momentum tensor for the given metric are

$$\bar{T}_t^t = - \left(\frac{1}{2\tau\Gamma} \dot{\psi}^2 + \frac{\psi'^2}{2A^2} \right) = -\bar{T}_y^y, \quad (57)$$

$$\bar{T}_i^i = \frac{1}{2\tau\Gamma} \dot{\psi}^2 - \frac{\psi'^2}{2A^2}$$

and

$$\bar{T}_y^t = -\frac{\dot{\psi}\psi'}{\tau\Gamma}.$$

To obtain the solutions for the scalar field, let us assume that

$$\psi(t, y) = \psi_1(t) + \psi_2(y). \quad (58)$$

From the ty -component of the field equations, we obtain

$$\dot{\psi}_1 = k \frac{\dot{A}}{A} \quad \text{and} \quad \psi'_2 = -\frac{3}{2k} \frac{\Gamma'}{\Gamma},$$

where k is an arbitrary constant. Consequently, the solution for the bulk scalar is given by

$$\psi(t, y) = k \int \frac{1}{A} \frac{\partial A}{\partial t} dt - \frac{3}{2k} \ln \Gamma + \text{constant}. \quad (59)$$

As a toy model if we assume that

$$A(t, y) = A_1(t) \times A_2(y), \quad (60)$$

then the result is

$$\psi(t, y) = k \ln A_1 - \frac{3}{2k} \ln \Gamma + \text{constant}. \quad (61)$$

However, we must remember that the assumption (60) is made only to obtain a clear picture of the nature of solution for a specific case, at the expense of the general character of the spacetime.

Equating the coefficient of $\frac{1}{A^2}$ on both sides of the yy -component of the field equations, we find that for a non-zero value of $\frac{\Gamma'}{\Gamma}$ we require

$$\Gamma(y) = C_1 e^{\pm cy}. \quad (62)$$

Considering a radiative brane with τ as a power-law function, for which

$$R(t) \sim \sqrt{t}, \quad (63)$$

$$\text{and} \quad \tau(t) \sim t^m, \quad (64)$$

we obtain

$$A_1(t) \sim t^{m(m+1)/2}, \quad (65)$$

$$A_2(y) \rightarrow 1. \quad (66)$$

To obtain a physically meaningful solution, $A_1(t)$ must be a decreasing function of time, which is true if we choose $m = -\frac{1}{2}$. This choice of m is based on the results obtained earlier [39].

Case 2:

Let us now consider $f(t, y) = 2bty$, where b is a dimensional parameter. In this case we have

$$\bar{G}_{ty} = 3b \left(\frac{t\dot{A}}{A} - 1 \right) \quad (67)$$

and

$$\bar{T}_{ty} = -\dot{\psi}\psi'. \quad (68)$$

Once again assuming a solution of the type given by (58), we obtain the following relations for the scalar field ψ :

$$\dot{\psi}_1 = -k_1 \left(\frac{t\dot{A}}{A} - 1 \right) \quad \text{and} \quad \psi'_2 = \frac{3b}{k_1},$$

for which, the solution is of the type

$$\psi(t, y) = -k_1 \left(\int \frac{t}{A} \frac{\partial A}{\partial t} dt - t \right) + \frac{3by}{k_1} + \text{constant}, \quad (69)$$

the exact solution being obtainable only when $A(t, y)$ is explicitly defined.

B. A scalar field with potential in the bulk

Let us now assume that the 5D bulk includes a dilaton-like background scalar field $\psi(t, y)$ with self-interaction scalar potential $V(\psi)$ [22, 37]. The energy-momentum tensor for a minimally coupled scalar field with potential is of the type

$$\bar{T}_{IJ}^{scalar} = -\partial_I \psi \partial_J \psi + g_{IJ} \left(\frac{1}{2} g^{KL} \partial_K \psi \partial_L \psi + V(\psi) \right). \quad (70)$$

The equation for the bulk scalar field is given by

$$\square_{(5)} \psi + \frac{\partial V}{\partial \psi} = 0. \quad (71)$$

With the assumed bulk metric (5), the scalar field equation of motion reduces to the following form:

$$e^{-2f} \ddot{\psi} - \frac{1}{A^2} \psi'' + e^{-2f} \left(2\dot{f} + \frac{3\dot{R}}{R} + \frac{\dot{A}}{A} \right) \dot{\psi} + \frac{1}{A^2} \left(-4f' + \frac{A'}{A} \right) \psi' = -\frac{\partial V}{\partial \psi}, \quad (72)$$

while the Einstein equations in the bulk are

$$\frac{3}{e^{2f}} \left(\frac{\dot{R}^2}{R^2} + \frac{2\dot{R}\dot{f}}{R} + \dot{f}^2 + \frac{\dot{f}\dot{A}}{A} + \frac{\dot{R}\dot{A}}{R\dot{A}} \right) - \frac{3}{A^2} \left(2f'^2 + f'' - \frac{f'A'}{A} \right) = - \left(\frac{\dot{\psi}^2}{2e^{2f}} + \frac{\psi'^2}{2A^2} - V + \bar{\Lambda} \right), \quad (73)$$

$$\dot{f}' - \frac{\dot{A}}{A} f' = \frac{\dot{\psi}\psi'}{3}, \quad (74)$$

$$\frac{3}{e^{2f}} \left(\frac{\ddot{R}}{R} + \frac{\dot{R}^2}{R^2} + \frac{3\dot{R}\dot{f}}{R} + \dot{f}^2 + \ddot{f} \right) - \frac{6f'^2}{A^2} = \frac{\dot{\psi}^2}{2e^{2f}} + \frac{\psi'^2}{2A^2} + V + \bar{\Lambda}, \quad (75)$$

$$\frac{1}{e^{2f}} \left(\frac{2\ddot{R}}{R} + \frac{\dot{R}^2}{R^2} + \frac{4\dot{R}\dot{f}}{R} + \dot{f}^2 + 2\ddot{f} + \frac{\dot{f}\dot{A}}{A} + \frac{2\dot{R}\dot{A}}{R\dot{A}} + \frac{\ddot{A}}{A} \right) - \frac{3}{A^2} \left(2f'^2 + f'' - \frac{f'A'}{A} \right) = \frac{\dot{\psi}^2}{2e^{2f}} - \frac{\psi'^2}{2A^2} + V + \bar{\Lambda}. \quad (76)$$

Equation (72) is not independent and can be derived from the Einstein equations.

In our model, as in [22], we consider that the brane universe undergoes a natural inflation. The scalar field with an effective potential $V(\psi)$ appears in the bulk on account of the dynamics of the bulk gravitational field. As the universe inflates, the potential decreases and finally becomes negligible, so that the spatially flat universe is recovered in the low energy limit, with the bulk cosmological constant settling down to a stable value. However, during the inflationary phase, the bulk cosmological constant was presumably time-varying. All these considerations make the problem very hard to tackle, especially by analytical methods, since neither the potential, nor the bulk cosmological constant, can be tuned to zero. Attempts have been made to address these type of complicated problems with the help of numerical methods. In spite of that we choose to proceed analytically and see how much information we can extract from this analysis.

1. Solutions

Assuming a solution of the type (58), and using the relations between the coefficients of $\frac{1}{A^2}$ on both sides of the third Einstein equation (75) leads us to the following possibility:

$$\psi'_2 = 2i\sqrt{3}f'. \quad (77)$$

Using this result in the second Einstein equation (74), we arrive at the relation

$$\dot{f}' = k_2 f' \quad \text{where} \quad k_2 = \frac{\dot{A}}{A} + \frac{2i}{\sqrt{3}} \dot{\psi}_1, \quad (78)$$

so that

$$\dot{\psi}_1 = \frac{\sqrt{3}}{2i} \left(k_2 - \frac{\dot{A}}{A} \right). \quad (79)$$

Therefore the form of $\psi(t, y)$ in general will be given by

$$\psi(t, y) = \frac{\sqrt{3}}{2i}k_2t - \frac{\sqrt{3}}{2i} \int \frac{1}{A} \frac{\partial A}{\partial t} dt + 2i\sqrt{3} \int \frac{\partial f}{\partial y} dy + \text{constant}. \quad (80)$$

If $A(t, y)$ is of the type (60), then

$$\psi(t, y) = \frac{\sqrt{3}}{2i}(k_2t - \ln A_1) + 2i\sqrt{3} \int \frac{\partial f}{\partial y} dy + \text{constant}. \quad (81)$$

Analysing the above equations we find that the variation of the scalar field along the extra dimension depends on the variation of the warping function along the same direction. Further, the time-variation of the scalar field is related to the time-variation of the extra-dimensional scale factor. It appears that the time-variation of the warping function may be related to the time-variation of the extra-dimensional scale factor in most cases. This picture is also physically consistent, since these time-variations are expected to dominate only during the early phases of evolution, for which these must be functions decaying with time.

We must remember that, along with this we have a time-varying bulk cosmological constant and an effective potential $V(\psi)$, which also decays with time. Since $\psi = \psi_1 + \psi_2$, we can write $V(\psi) = V(\psi_1) + V(\psi_2)$. At points where the space-variation of the potential is negligible, viz. at sufficient distances from the source, we can neglect the ψ_2 part and assume that $V(\psi) = V(\psi_1)$. However, close to the source, this assumption is invalid during the earlier epochs of time, when neither ψ nor $V(\psi)$ is negligible. As a toy model we therefore consider a given form of potential $V(\psi)$, like the sine-Gordon potential, which dynamically generates the brane universe in the form of a thick brane in the background warped geometry from nothing. The universe inflates asymptotically at the Hubble rate $H(l)$, where, l is the AdS radius

$$l = \sqrt{-\frac{6}{\bar{\Lambda}}}. \quad (82)$$

The associated brane tension σ will be of the type

$$\sigma = \sigma_c T(t, y), \quad (83)$$

where σ_c is the critical tension that reproduces the original RS braneworld. We assume that the bulk potential varies very slowly in space and time, so that the braneworld is created automatically from the dynamics of the space-time. The sine-Gordon potential is given by

$$V(\psi) = B_1 + B_1 \cos \frac{2\psi}{B_2}. \quad (84)$$

Here B_2 must be nonzero. For simplicity, we choose $B_2 = 1$. Thus B_1 gives the value of potential when ψ is zero i.e. $V(0) = 2B_1$. Under this condition we have $\psi_1 = -\psi_2$. We consider the situation where the scalar field ψ is close to zero. In the lowest order, we can put $\psi = 0$, so that the bulk is AdS with an effective cosmological constant $\bar{\Lambda}_{eff}$, which is given by

$$\bar{\Lambda}_{eff} = \bar{\Lambda} + 2\bar{\kappa}^2 B_1, \quad (85)$$

from which we have

$$B_1 = \frac{\bar{\Lambda}_{eff} - \bar{\Lambda}}{2\bar{\kappa}^2}, \quad (86)$$

with $|\bar{\Lambda}| > 2\bar{\kappa}^2 B_1$, so that $\bar{\Lambda}_{eff} < 0$, thereby preserving the AdS nature of the bulk. The effective AdS radius is now

$$l_{eff}^2 = \frac{6}{|\bar{\Lambda}_{eff}|}, \quad (87)$$

with $\bar{\kappa}^2 = \frac{|\bar{\Lambda}|}{6}$.

To find the expression for the bulk cosmological constant, we subtract (73) from (76) and arrive at the relation

$$2\bar{\Lambda} = \frac{1}{e^{2f}} \left(2 \left(\dot{H} - H\dot{f} - \dot{f}^2 + \ddot{f} - \frac{\dot{f}\dot{A}}{A} \right) - H\frac{\dot{A}}{A} + \frac{\ddot{A}}{A} - \dot{\psi}_1^2 \right) \quad (88)$$

where $H = \frac{\dot{R}}{R}$. Assuming $2\dot{f} = \frac{\dot{A}}{A}$, we get

$$2\bar{\Lambda} = \frac{1}{e^{2f}} \left(2 \left(\dot{H} - 2H\dot{f} - \dot{f}^2 + 2\ddot{f} \right) - \dot{\psi}_1^2 \right). \quad (89)$$

Therefore the AdS radius and the bulk cosmological constant are both time-dependent and also dependent on the extra-dimensional coordinate.

2. Cosmological interpretations:

Although the bulk spacetime (5) is not conformally flat, its 4D part is found to possess a vanishing Weyl tensor and therefore is conformally flat. The Einstein tensor of the bulk is diagonal only under the condition (38), but it is always diagonal for its corresponding 4D part. In general, the Ricci scalar for both the bulk and the 4D part are negative, indicating that both are AdS, except under some special circumstances. The constraint for the 4D part to be AdS is obtained as follows:

$$\dot{f}^2 + 2H^2 > |\ddot{f} + 3H\dot{f} + \dot{H}|, \quad (90)$$

while for the full 5D spacetime, we must have

$$3(\dot{f}^2 + 2H^2) > \left| 3 \left(\ddot{f} + 3H\dot{f} + \dot{H} + \frac{H\dot{A}}{A} \right) + \frac{2\dot{f}\dot{A}}{A} + \frac{\ddot{A}}{A} \right|, \quad (91)$$

$$\text{and} \quad \frac{5\dot{f}'^2}{2} > \left| f'' - \frac{f'A'}{A} \right|. \quad (92)$$

The bulk pressure is isotropic along the brane, but is different along the extra dimension. However, the 4D part has isotropic pressure. The 4D part of this 5D bulk represents a universe with pure radiation if

$$R\ddot{f} + \dot{R}\dot{f} + \ddot{R} = 0. \quad (93)$$

In that case $R \sim \sqrt{t}$, and (93) reduces to $4\ddot{f}t^2 + 2\dot{f}t - 1 = 0$.

VI. EFFECT OF PLACING THE BRANE AND LOCALIZATION OF FIELDS

Having determined the bulk solutions, we now consider the effect of placing the single 3-brane in the bulk spacetimes considered above. This amounts to evaluating the brane tension σ in terms of the parameters appearing in the expression for the bulk fields. For that we need to introduce the contribution of the 3-brane into the bulk action, as follows:

$$S_b = \int d^5x \sqrt{g} V_b(\psi) \frac{\delta(y - y_0)}{\sqrt{g_{yy}}}. \quad (94)$$

The delta function appearing above will be peaked at the location of the brane ($y = y_0$). The Einstein equations and hence the corresponding energy-momentum tensor for the brane-bulk system will get modified due to the appearance of the delta function term $\sigma\delta(y)$. Integrating the scalar field equation for the massless scalar at the location of the brane and using the Einstein equations, along with the constraint $\psi'_2 = -\frac{3}{2k}\frac{\Gamma'}{\Gamma}$, we obtain

$$\sigma \sim \sigma_c. \quad (95)$$

However, for an inflating brane, the departure of σ from σ_c is not negligible [33].

We now briefly discuss the nature of localization of the fields [10–12, 24, 50] for the spacetimes considered above. To explore the issue of localization of the various types of matter fields, the standard model fields are assumed to be distributed evenly over the entire bulk. The behavior of the KK modes along the extra dimension then indicate whether the fields are localized on the brane or not. Although the scalar (spin 0) and gravity (spin 2) fields can be localized on the positive tension brane in the RS background, it is generally not the case for the massless fermions. The massless chiral fermions can be localized on the positive tension RS brane in presence of Yukawa couplings with the bulk scalar. With an increasing warp factor, both the chiral zero modes may be confined to the brane, even in

the absence of any coupling with the bulk field. However, all the standard model fields may not be localizable on a single brane through gravitational interactions only, for other types of scalar fields and warp factors.

To check for graviton localization, the spectrum of linearized tensor perturbations (restricted to 4D around the classical solutions of the warped spacetime), is analysed. The transverse traceless (TT) modes of the gravitational fluctuations represent the gravitons on the brane. These zero modes are analysed under the RS gauge conditions. In the typical RS background, the graviton zero mode is found to be localised and normalisable. In the braneworld models with massless bulk scalars and having a sine-Gordon potential, localized zero modes appear on the brane. With the flat 3-brane of the original RS setup being replaced by a dynamical brane in the generalized background as the one considered by us, the zero-mode graviton fluctuation is not guaranteed to be localized on the brane for an AdS bulk. The massless graviton wave function turns out to be non-normalizable once the Z_2 symmetry is relaxed, and the 4D Newton's constant may not be finite. The choice of bulk potential, the nature of coupling and the type of warp factor is crucial for achieving a normalizable zero mode.

VII. SUMMARY AND DISCUSSIONS

In this paper, we have considered five-dimensional warp product space-times with a time-dependent warp factor and a non-compact fifth dimension. The warp factor reflects the confining role of the bulk cosmological constant to localize gravity at the brane through the curvature of the bulk. Since this process of localization may include some time-dependence during a particular epoch of time, we have considered a generalized warp factor in the form of $e^{2f(t,y)}$, which depends both on time as well as on the extra-dimensional coordinate. The extra-dimensional scale factor is also a function of time and of the extra coordinate, so that we are dealing with a very general type of bulk. The energy eigen values are evaluated and the condition for the WEC and NEC to hold in the bulk are discussed. Coupling these conditions with the conservation equations, we have been able to comment on the nature of matter-energy tensor for this bulk and the type of warping function. Assuming a definite correlation between the rate of variation of the warping function and the normalized rate of variation of the extra-dimensional scale factor, we have found that the warping function turns out to be of the product type for a radiative brane.

At low energies, when gravity and particles remain confined to the brane, we have found that the extra-dimensional scale factor depends on the variation of the warping function along the extra dimension. In general, the bulk is not conformally flat, although it can be so if the extra-dimensional scale factor is only a function of time. For a particular choice of the warping function such that the 5D metric may be conformally flat, the Einstein tensor for the bulk and its 4D part are diagonal and the respective Ricciscalars are generally negative. The braneworld represents an universe in an epoch of uniform acceleration. For a stabilized bulk of constant curvature, our model represents the usual RS scenario. Finally, at high energies, we have considered the most general warped metric for the bulk to be sourced by scalar fields living in the bulk. For an AdS bulk sourced by a non-self-interacting massless scalar field, we have determined the solutions for the scalar field, the warp factor and the extra-dimensional scale factor. Next, considering a dilaton-like scalar field with a sine-Gordon type of potential, we have determined the effective cosmological constant of the AdS bulk in terms of the bulk potential and have found the solution for the scalar field. The background geometry obtained in this case is described by a negative, non-constant Ricciscalar. The energy-momentum tensor, arising from this geometry may violate the strong energy condition depending on the nature of warping, as well as on the effect of the extra dimension, but the weak and the null energy conditions will be satisfied. The bulk in general will be anisotropic, but the corresponding four-dimensional part is isotropic. Although the brane tension is close to the critical value which reproduces the original RS braneworlds, yet the difference is not negligible. For this inflating braneworld model, the zero mode graviton fluctuation may not be localized on the brane.

A detailed analysis of the localization of gravitons and other gauge fields will be presented in a separate report, along with the corresponding stability analysis, the nature of bulk singularities and other related aspects.

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